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A STUDY OF CONTACT INTERACTION OF TWO-LAYER SHELLS

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In solving problems of the stress-strain state of multilayer shell structures it is necessary to take into account the interaction conditions of adjacent layers. Most studies have dealt with problems of shell mechanics under the assumption that the mechanical contact of the layers is ideal. The current state of and approaches to solving the problems of the theory of multilayer shells with possible delamination zones are reflected in [1].

The purpose of this paper is to find an approach for analysis of two-layer shells of revolution taking into account the possible one-sided character of contact interaction concerning contact problems [2, 3]. The interaction of orthotropic layers over a contact region which was known in one direction was considered in [4]. The boundary surfaces of the layers were modeled by adhesive laminae with different bed moduli, thicknesses, and pull-off character of deformation. In this paper, the approach [4] is extended to the class of two-dimensional contact problems. As a result, the solution proposed enables us to determine the unknown contact region between the layers in two directions $\Omega_+(s,\theta)$, the distribution of contact pressure $q(s,\theta)$, and the stressed state of a shell as a function of contact pressure and external loading.

We assume that each layer of the two-layer shell is described by its own differential equations. It is also assumed that the layers obey geometrically and physically the conditions of the linear theory of shells, there are no stresses or strains, the temperature field is constant, and contact between the layers occurs without friction or slipping. When considering equidistant layers with clearance, the clearance must not exceed the shell thickness, since a geometrically linear theory is used. The starting system of equilibrium equations of shell layers interacting with each other has the form [5]

$$L^{(i)}\mathbf{Y}^{(i)} = \mathbf{f}^{(i)} - (-1)^i q\lambda M, \quad i = 1, 2, \quad \lambda(s,\theta \in \Omega_+) = 1, \quad \lambda(s,\theta \notin \Omega_+) = 0, \tag{1}$$

where i is the shell-layer number, L is a matrix differential operator, $\mathbf{Y}$ is the unknown vector of resolvent functions, $\mathbf{f}$ is the vector function of the external distributed load, M is a column matrix whose element corresponding to the equilibrium equation in projection on a vector normal to the surface Ω_+ is equal to unity with the other elements equal to zero, and s and θ are meridional and circumferential coordinates on a shell surface.

For many shell models, a governing system of equations written in a system of principal curvatures α and β has the form [6]

$$\frac{\partial \mathbf{Y}}{\partial \alpha} = \sum_{m=0}^{4} A_m(\alpha,\beta)\,\frac{\partial^m \mathbf{Y}}{\partial \beta^m} + \mathbf{f}(\alpha,\beta). \tag{2}$$

Here, matrices A depend on geometrical and mechanical characteristics, while the dimension of vector $\mathbf{Y}$ and the order of the equations m depend on shell model chosen.

Taking (1) and (2) into account, we write a system of differential equations describing the contact interaction of layers of a shell of revolution as follows

$$\frac{\partial \mathbf{Y}^{(i)}}{\partial s} = \sum_{m=0}^{4} A_m^{(i)}(s,\theta)\,\frac{\partial^m \mathbf{Y}^{(i)}}{\partial \theta^m} + \mathbf{f}^{(i)}(s,\theta) - (-1)^i q\lambda M, \quad i = 1, 2. \tag{3}$$

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System (3) is to be supplemented by boundary conditions for two contours of each layer $s = s_0$ and $s = s_L$ [6]

$$B_1^{(i)}\mathbf{Y}^{(i)}(s_0) = \mathbf{b}_1^{(i)}, \quad B_2^{(i)}\mathbf{Y}^{(i)}(s_L) = \mathbf{b}_2^{(i)}, \quad i = 1, 2$$

(B are $n \times n/2$ specified matrices, n is the dimension of $\mathbf{Y}$, and $\mathbf{b}$ are specified vectors) and also the condition that one layer does not penetrate into another [5].

Since q appearing in system (3) is a function of two coordinates, this system describes the class of two-dimensional contact problems. We will use here the classical theory of anisotropic nonhomogeneous shells [6, 7].

One feature of the problems to be solved is investigation of the size of the unknown contact region $\Omega_+(s, \theta)$, that is, problems with a free boundary, where the following condition must be satisfied

$$q(s, \theta \in \Omega_+) \geqslant 0. \tag{4}$$

We supplement unknown contact region Ω_+ by delamination region Ω_- to a certain specified region Ω completely containing the free boundary. We divide the region Ω into equal parts in circumferential and meridional directions. Taking into account the small size of the elements obtained, we assume that interaction of shell layers can be represented by unknown forces X applied at a number of points of region Ω; consequently,

$$\Omega = \sum_j^K \sum_i^N F_{ij},$$

where $F = a_\theta \times a_s$ is the area of a contact element, a_θ and a_s are the linear dimensions in the circumferential and meridional directions, and N and K are the numbers of elements along the circumference and meridian, respectively.

Such discretization of region Ω [2, 3] enables problems of layer interaction to be reduced to well-known methods of analysis of the stress-strained state of orthotropic shells of revolution based on reduction of a boundary-value problem to a set of Cauchy problems using S. K. Godunov's orthogonalization [6].

Contact load (load due to interaction of the layers) can be represented by a certain number of unknown absolutely rigid connections, which are determined using the force method of structural mechanics. A canonical system of equations describing the contact condition for the layers of a ring cut out of the shell has the form

$$\begin{aligned}
&\sum_{i=1}^{N} \delta_{1i}^{(1)} X_i^{(1)} + \underline{\sum_{i=1}^{N} \delta_{1i}^{(2)} X_i^{(2)}} + \ldots + \underline{\sum_{i=1}^{N} \delta_{1i}^{(k)} X_i^{(k)}} + D X_1^{(1)} + \Delta_{P1}^{(1)} + \Delta_{R1}^{(1)} = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\sum_{i=1}^{N} \delta_{Ni}^{(1)} X_i^{(1)} + \underline{\sum_{i=1}^{N} \delta_{Ni}^{(2)} X_i^{(2)}} + \ldots + \underline{\sum_{i=1}^{N} \delta_{Ni}^{(k)} X_i^{(k)}} + D X_N^{(1)} + \Delta_{PN}^{(1)} + \Delta_{RN}^{(1)} = 0.
\end{aligned} \tag{5}$$

Here, δ_{ij} is the displacement in the basic system in the direction of the ith connection due to a unit force acting in the direction of the omitted jth connection, which is Δ_{Ri} is the value of clearance in the direction of the ith connection in problems of disconnected layers, Δ_{Pi} is the displacement in the direction of the ith connection, which is caused by a specified external load acting on shell [3], D is an operator relating the reactive force at the ith point on the surface of possible elastic gaskets (or elastic properties of the microgeometry of the contacting surfaces) between the layers to its displacement [in the case of the Winkler model we have $D = (CF)^{-1}$, where C is the bed modulus], and $N = 2t_\theta(a_\theta)^{-1}$ ($2t_\theta$ is the length of Ω in the circumferential direction), the terms which are not underlined define the coupling of the first ring for one-dimensional contact problems [4], and underlined ones define the influence of the remaining $K - 1$ rings on the first ring. The flexibility of a contact shell element (influence function) δ_{ij} appearing in (5) is constructed numerically by integrating the equations for each shell layer when unit step forces are distributed over each element [3].

Taking into account the vertical symmetry of the problem with respect to section $\theta = 0$, we write system (5) in matrix form

$$[H_{11}H_{12}\dots H_{1K}]\begin{bmatrix}\{X^{(1)}\}\\ \{X^{(2)}\}\\ \dots\\ \{X^{(k)}\}\end{bmatrix} + \{\Delta_P^{(1)}\} + \{\Delta_R^{(1)}\} = 0,$$

where $X_1^{(1)}\dots X_M^{(1)}\dots X_1^{(k)}\dots X_M^{(k)}$ are the vectors of the unknown forces of contact interaction ($M = N/2$); $\{\Delta_P^{(1)}\} = \{\Delta_{P1}^{(1)}\dots\Delta_{PM}^{(1)}\}$ is the vector of displacements of the first ring due to external forces P; $\{\Delta_R^{(1)}\} = \{\Delta_{R1}^{(1)}\dots\Delta_{RM}^{(1)}\}$ is the vector of clearances; and

$$H_{mp} = \begin{bmatrix} w_{1mp}+w_{2mp}+\delta^*_{mp}D & w_{2mp}+w_{3mp} & w_{3mp}+w_{4mp} & \dots & w_{Mmp}+w_{M+1,mp} \\ & w_{1mp}+w_{4mp}+\delta^*_{mp}D & w_{2mp}+w_{5mp} & \dots & w_{M-1,mp}+w_{M-2,mp} \\ \dots\dots & \dots\dots & \dots\dots & \dots & \dots\dots \\ \text{(symmetrically)} & & & \dots & w_{1mp}+w_{2M,mp}+\delta^*_{mp}D \end{bmatrix}. \quad (6)$$

In matrix (6), trivariate indexing nmp is introduced ($n = 1\text{–}2M$ enumerates elements along the circumference, $m = 1\text{–}K$ enumerates rings, and $p = 1\text{–}K$ enumerates the ring, on which a unit force acts), δ^*_{mp} is the Kronecker delta, and $w_i = \delta_{1i}$ and $\delta_{ij} = \delta_{ji}$ ($i = 1, \dots, N$).

In the same way we construct coupling conditions for all K rings and reduce them to the system

$$\begin{bmatrix}[H_{11}] & [H_{12}] & \dots & [H_{1k}]\\ [H_{21}] & [H_{22}] & \dots & [H_{2k}]\\ \dots & \dots & \dots & \dots\\ [H_{k1}] & [H_{k2}] & \dots & [H_{kk}]\end{bmatrix}\begin{bmatrix}\{X^{(1)}\}\\ \{X^{(2)}\}\\ \dots\\ \{X^{(k)}\}\end{bmatrix} + \begin{bmatrix}\{\Delta_p^{(1)}\}\\ \{\Delta_p^{(2)}\}\\ \dots\\ \{\Delta_p^{(k)}\}\end{bmatrix} + \begin{bmatrix}\{\Delta_R^{(1)}\}\\ \{\Delta_R^{(2)}\}\\ \dots\\ \{\Delta_R^{(k)}\}\end{bmatrix} = 0. \quad (7)$$

System (7) consists of $K \times M$ equations and contains $K \times M$ unknowns. Consequently, equations defining contact interactions (7) corresponding to the first approximation can be written as a system of linear algebraic equations

$$[\widehat{A}]_j\{\widehat{X}\}_j = \{\widehat{B}_P\}_j + \{\widehat{B}_R\}_j, \quad j = 1, \quad (8)$$

where vector $\{\widehat{B}_P\}_j$ on the right-hand side is found by integrating a shell layer, to which an external load is applied.

In solving problems with two-sided connections (when delamination does not occur) system (8) defines completely the vector of contact pressures $\{q\} = X_iF^{-1}$, $i = 1, \dots, M$. In solving problems with one-sided connections [8], condition (4) must be satisfied and it is necessary to find a resolvent system such that contact pressure is positive on the contact regions of the shell layers and equal to zero in possible zones of delamination. To determine those unknowns which are to be eliminated, we use the method of successive approximations, which is usually employed for solving contact problems numerically [5, 9], the essence of which is to construct a jth approximation using the preceding $(j-1)$th approximation on the condition that connections are absent at those regions where $X < 0$.

In solving problems for layers connected by glue lamina when the working ability of the glue joint under stretching is limited, the problem is solved as in the case of one-sided connections; however the condition of breaking of nonworking connections has the form $X_i/F < \sigma_g$ (σ_g is the cohesive strength of the glue joint).

Determination of the stress-strain state of the shell under consideration comes down to integration of the equations for each shell layer for external load P and the contact load determined in [3]

$$\{Q_\Sigma\} = \{P\} + \{Q\}, \quad \{Q\}_j \in \Omega_+, \quad \{P\} \in \Omega_0$$

(Ω_0 is the shell surface on which the vector of external load is applied).

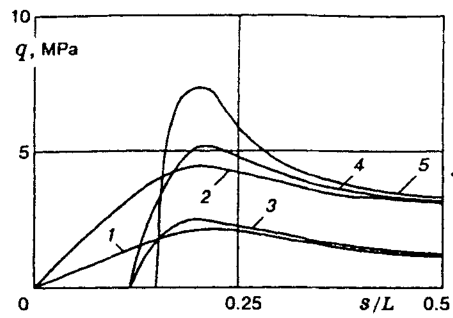

Fig. 1

TABLE 1

K	M: 1	2	3	4	5	6
	q, MPa					
1	0.7373	0	0	0.6721	0	0
2	2.6450	0	0	0.2469	0	0
3	5.2840	0	0	0.4716	0	0
4	8.8920	0	0	0.6646	0	0
5	12.33	0	0	0.7784	0	0
6	12.33	0	0	0.7752	0	0
7	8.884	0	0	0.6586	0	0
8	5.273	0	0	0.4640	0	0
9	2.636	0	0	0.2413	0	0
10	0.7352	0	0	0.6580	0	0

Since the function of the total load $\{Q_\Sigma\}$ defined on a set of contact elements is an even periodic function with period $2M$, it can be expanded into a Fourier cosine series [2, 10]. The expansion coefficients for the jth ring have the form

$$a_k^{(j)} = \frac{2}{M}\left[\frac{1}{2}q_{\Sigma 0}^{(j)} + \sum_{i=1}^{M-1} q_{\Sigma i}^{(j)} \cos\frac{\pi k}{M} i + \frac{1}{2}(-1)^k q_{\Sigma M}^{(j)}\right], \quad b_k^{(j)} = 0.$$

As an example illustrating the above method we have considered the interaction of coaxial cylindrical shells of the same length ($L = 0.1524$ m) and thickness ($h = 0.254 \cdot 10^{-2}$ m). This problem of interaction of separated shells (layers) has been treated in [5, 11]. Two shells (the radius of the inner shell is $R_2 = 0.0762$ m) are fixed rigidly with clearance $\Delta_R = 0.0127 \cdot 10^{-2}$ m. The inner shell is loaded with pressure $p = 20.67$ MPa. The shell material is isotropic and characterized by Young's modulus $E = 2.1 \cdot 10^5$ MPa and Poisson's ratio $\nu = 0.3$. The division angle for the circumference is $\Delta\theta = 30°$.

In this problem, the number of divisions of the shells in the circumferential direction N is of no principal significance. However, an increase in the number of divisions of the shells in the meridional direction K leads to a better illustration of the edge effect. A solution has been obtained for two values of contact-element length in the meridional direction a_s. In the first case, half of region Ω is approximated by 60 contact elements with area $F = a_\theta a_s = 3.99 \cdot 1.524 = 6.08$ cm^2. Hence, the order of the matrix in (6) is $M = 6$ and that in (7) is $K = 10$. Since the external load and clearance are axisymmetrical and the problem is symmetrical with respect to section $s = 0.5L$, the contact pressure q is axisymmetrical and given for a quarter of region Ω.

In Fig. 1 we show the contact pressure distribution along the meridian of the shell. Curves 1 and 2 correspond to calculations performed for bed modulus $C = 100C_r$ and $500C_r$ (the value $C_r = 10^8$ N/m^3

TABLE 2

	M					
	1	2	3	4	5	6
K	q, MPa					
1	0.734	−0.221	−0.217	0.655	−0.0003	−0.0068
2	2.634	−0.8757	−0.6981	0.2404	−0.008	−0.023
3	5.266	−1.892	−1.148	0.4618	−0.0286	−0.0404
4	8.870	−3.026	−1.445	0.6542	−0.05497	−0.054
5	12.31	−3.822	−1.594	0.767	−0.07451	−0.062

corresponds to elastic properties of vacuum rubber [9]). Curves 3 and 4, which refine curves 1 and 2, were obtained using a smaller division step along the meridian, the area of contact element being $F = a_\theta a_s = 3.99 \cdot 0.762 = 3.04$ cm^2 ($M = 6$ and $K = 20$). Curve 5 represents numerical solution [5]. A solution of the problem close to the solution obtained here has been found analytically in [11].

Distribution of the contact pressure q along the circumference ($M = \overline{1, 6}$) and the meridian ($K = \overline{1, 10}$) for the case of nonaxisymmetrical loading of a two-layer shell (without clearance) is given in Table 1. External loading is represented by the force $P = 10{,}000$ distributed over the area $F_p = a_\theta 2a_s$ centered at $s = L/2$ and $\theta = 0$ and applied to the external layer. A solution is found for the case of $F = 6.08$ cm^2 and $C = C_r$. An indirect confirmation of the correctness of our calculations is that the solution is symmetric with respect to section $s = 0.5L$. The value of the bed modulus C chosen for calculations which imitates the microroughness of the surfaces in contact and also the elastic properties of the glue lamina and the cohesive strength of the glue joint $\sigma_g = 0$ provides for contact at the point where the force P is applied ($K = 5,\ 6$ and $M = 1$) and the appearance of complicated delamination zones.

The distribution of q (interlaminar stress) for the case where there is no delamination is given in Table 2. It follows from the above results that the case considered here can be realized if the cohesive strength of the glue joint is $\sigma_g > 3.822$ MPa ($K = 5$ and $M = 2$).

Thus, the approach in question and its realization using stable numerical methods enable us to determine contact stresses in layered shells, to determine delamination zones unknown in two directions, and to take into account nonhomogeneous adhesive laminae, the one-sided character of the interaction of layers, and material anisotropy.

REFERENCES

1. E. I. Grigolyuk, E. A. Kogan, and V. I. Mamai, "Deformation problems of thin-walled laminated structures with delaminations," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 2 (1994).
2. I. G. Emel'yanov, "Numerical analysis of contact interaction of cylindrical shells," *Prikl. Mekh.*, **23**, No. 6 (1987).
3. A. T. Vasilenko and I. G. Emel'yanov, "Contact problem of a thin cylindrical shell resting on a round base," *Probl. Prochn.*, No. 6 (1990).
4. A. T. Vasilenko and I. G. Emel'yanov, "Investigation of contact interaction of layers in shells of revolution," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 3 (1994).
5. B. Ya. Kantor, *Contact Problems of Nonlinear Theory of Shells of Revolution*, [in Russian], Naukova Dumka, Kiev (1990).
6. Ya. M. Grigorenko and A. T. Vasilenko, "Methods of Calculation of Shells," in: *Theory of Shells of Variable Stiffness* [in Russian], Naukova Dumka, Kiev (1981), vol.4.
7. S. A. Ambartsumyan, *General Theory of Anisotropic Shells* [in Russian], Nauka, Moscow (1974).
8. I. M. Rabinovich, *Problems of Theory of Static Calculation of Constructions with One-Sided Connections* [in Russian], Stroiizdat, Moscow (1975).

9. V. I. Mossakovskii, V. S. Gudramovich, and E. M. Makeev, *Contact Interaction of Elements of Shell Structures* [in Russian], Naukova Dumka, Kiev (1988).
10. R. W. Hamming, *Numerical Methods for Scientists and Engineers*, McGraw-Hill, New York (1962).
11. S. V. Kulkarni and D. Frederick, "The contact problem of two coaxial cylindrical shells," *Int. J. Mech. Sci.*, **15**, No. 5 (1973).